WHAT IS ANCIENT MATHEMATICS?

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"In my opinion, it is not only the serious accomplishments of great and good men which are worthy of being recorded, but also their amusements."

Xenophon, *Symposium*

The title of this paper is a result of comments on earlier drafts by mathematicians: "This is not mathematics, this is history!" and by historians of mathematics: "This is not history, this is mathematics!" After some reflection, I came to the conclusion that the historians were right and the mathematicians were wrong–for example, I have found little difference between reading papers of Atle Selberg (1917–, Fields Medal 1950) and Archimedes (287–212 B.C.) (who both lived in Syracuse!). I believe that the mathematicians I spoke to were expressing a generally held belief that reading mathematical papers that are over a hundred years old is history of mathematics, not mathematics. Thus, the reconstruction of Heegner's solution to the class number one problem (1952) appeared in a mathematics journal [56] while a reconstruction of the missing portions of Archimedes's *The Method* (250 B.C.) appeared in a history journal [29].

To me, reading and proving results about a mathematical paper, whether it was written in 1950 A.D. or 250 B.C., is always mathematics, though the latter case might be called "ancient mathematics." At least as to Greece, this is accepted by some [30, p. 21]:

> Oriental mathematics may be an interesting curiosity, but Greek mathematics is the real thing. . . The Greeks, as Littlewood said to me once, are not clever schoolboys or "scholarship candidates," but "Fellows of another college." So Greek mathematics is "permanent," more perhaps even than Greek literature. Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not.

I am saying that ancient Greek mathematicians were in every essential way similar to modern mathematicians. In fact, some mathematicians might find more in common with Archimedes and Euclid than with many colleagues of their departments, and even reading the original Greek, a subject traditionally taught in High School [9], seems easier than understanding say, the proof that every semistable elliptic curve is modular [63].

Nineteenth century mathematicians dedicated much of their research to elementary Euclidean geometry. It is possible that some mathematicians of that era felt that the influence of the past was too great, as Felix Klein wrote [39, Vol. 2, p. 189]:

> Although the Greeks worked fruitfully, not only in geometry but also in the most varied fields of mathematics, nevertheless we today have gone beyond them everywhere and certainly also in geometry.

For whatever reason, geometers recently tend to distance themselves from Euclidean geometry. For example, the book *Unsolved Problems in Geometry* [16], part of a series on "unsolved problems in intuitive mathematics" does not have a section devoted to classical Euclidean geometry, and with few exceptions such as [10], articles on this subject are relegated to "low–brow" publications. Yet earlier in this century, Bieberbach, Hadamard, and Lebesgue all wrote books on elementary Euclidean geometry [13] [27] [45] and excellent books and articles on ancient mathematics are still being written [31] [59]. See [17] for further analysis of these issues.

In this paper, I will give an example of ancient mathematics by using techniques that Archimedes developed in his paper *The Method* to derive results that he proved in his paper *On Spirals*. I will try to present these in a way that Archimedes might understand [61], in particular, the diagrams are intended to conform to ancient Greek standards [47]. I will also indicate how ideas in these papers can lead to some surprising results (e.g., Exercise 4 below). The paper will include such exercises as may challenge the reader to understand concepts of Archimedes as he expressed them.

I have concentrated on the works of Archimedes because these are most similar to modern mathematical research papers, sharply focused on problems and their solution. By comparison, the works of Euclid read like a generic textbook; and so little is known about Euclid that it cannot be ruled out that he was actually a "consortium." Moreover, it seems likely that the works of Euclid are based on the efforts of earlier mathematicians [24] [40].[1]

The balance of the paper shows how a precise knowledge of ancient mathematics allows one to navigate in the sea of inaccuracies and misconceptions written about the history of mathematics. This also gives one perspective on cultural aspects of mathematics as it forces one to understand ideas of first rate mathematicians whose cultural background is very different from the present one. For example, it can help you read *The New York Times* [37]:

> "Alien intelligences may be so far advanced that their math would simply be too hard for us to grasp," [Paul] Davies said. "The calculus would have baffled Pythagoras, but with suitable tuition he would have accepted it."

Reading this paper should make it clear that Archimedes could have been Pythagoras's calculus tutor, thus refuting any notion that calculus was an unknown concept to ancient Greeks.

It is my hope that I can convince mathematicians that there are many interesting and relevant ideas to be uncovered in ancient Greek mathematics, and that it might be worthwhile to take a first hand look, being wary of popular accounts and secondary sources, this one included!

Extending Archimedes's Method

In 1906 the Danish philologist J.L. Heiberg went to Constantinople to examine a manuscript containing mathematical writing which had been discovered seven years earlier in the monastery of the Holy Sepulchre at Jerusalem. What he found was a 10th–century palimpsest, a parchment containing works of Archimedes that, sometime between the 12th and 14th centuries, had been partially erased and overwritten by religious

[1] Hence I question the curriculum of St. John's College which purports to educate its students by following an historical sequence of original sources. Its reading list also includes the ancient textbook [48].

text. Heiberg managed to decipher the manuscript [33] and found that it included a text of *The Method,* a work of Archimedes previously thought lost. (The story of the transmission of Archimedean manuscripts given in [18] reads like a chapter from *The Maltese Falcon*). Late bulletin: Heiberg's palimpsest was sold in auction to an unknown buyer for $ 2,000,000 on October 29, 1998 [52]. This gave scholars the opportunity to examine it for the first time in almost 100 years [51].

Heiberg's discovery had a significant impact on the understanding of ancient Greek mathematics, for two reasons. The first is the aim of the paper, summarized by Archimedes[2] [58, Vol. 2, p. 221]

> Moreover, seeing in you, as I say, a zealous student and a man of considerable eminence in philosophy, who gives due honour to mathematical inquiries when they arise, I have thought fit to write out for you and explain in detail in the same book the peculiarity of a certain method, with which furnished you will be able to make a beginning in the investigation by mechanics of some of the problems in mathematics. I am persuaded that this method is no less useful even for the proofs of the theorems themselves. For some things first became clear to me by mechanics, though they had later to be proved geometrically owing to the fact that investigation by this method does not amount to actual proof; but it is, of course, easier to provide the proof when some knowledge of the things sought has been acquired by this method rather than to seek it with no prior knowledge.

This is a radical divergence from all other extant Greek works, as T.L. Heath explains [6, Supplement, p. 6]:

> Nothing is more characteristic of the classical works of the great geometers of Greece, or more tantalising, than the absence of any indication of the steps by which they worked their ways to the discovery of their great theorems. As they have come down to us, these theorems are finished masterpieces which leave no traces of any rough–hewn stage, no hint of the method by which they were evolved... A partial exception is now furnished by the *Method*; for here we have a sort of lifting of the veil, a glimpse of the interior of Archimedes's workshop as it were.

The other surprising aspect of *The Method* is the revelation that Archimedes worked with *infinitesimals,* for example, "The triangle ΓZA is composed of the straight lines drawn in ΓZA," [58, Vol. 2, p. 227], "The cylinder, the sphere and the cone being filled by circles thus taken... " [8, Vol. 3, p. 91], see [1] [43]. As every mathematician knows, infinitesimals were reinvented by mathematicians such as Cavalieri (1598–1647) and Leibniz (1646–1716), see [2] [21]. Archimedes used them to compute the area and volumes of various geometrical figures including what he considered his greatest achievement:[3]

any cylinder having for its base the greatest of the circles in the sphere, and having its height equal to the diameter of the sphere, is one–and–a–half times the sphere,

a result he subsequently proved, *On the Sphere and Cylinder, I,* Corollary to Proposition 34 [58, Vol. 2, p. 125]. Archimedes understood that his method does not produce valid proofs due to its use of infinitesimals,[4] though

[2] Archimedes is addressing Eratosthenes of Cyrene (circa 284–194 B.C.), director of the library of Alexandria, famous for his accurate measurement of the circumference of the earth [14] and his sieve to compute prime numbers [48].

[3] Archimedes requested that a diagram of a sphere inscribed in a cylinder along with their proportion be placed on his grave, which Cicero reported finding in 75 B.C. when he was treasurer of Sicily [58, Vol. 2, p. 33].

[4] In *The Quadrature of the Parabola* Archimedes gave what he considered to be a rigorous proof using the mechanical method of a result conjectured in a similar way in *The Method,* but using infinitesimals.

it is unclear if the same is true of his successors. In any case, it is easy to make the arguments rigorous, given present knowledge. The basic ideas of *The Method* are still presented in contemporary calculus courses [35] [55, p. 709], and a physical model of Archimedes's argument has been built [25].

On the other hand, Archimedes's *On Spirals* is a masterpiece of rigorous mathematics. In this paper, Archimedes computes the area and tangent of a spiral, and, in doing so, derives much of the Calculus I curriculum, including related rates, limits, tangents, and the evaluation of Riemann sums. This is reflected by the fact that a number of contemporary Calculus texts outline the basic idea of Archimedes's computation of the area of a spiral [3, p. 3] [12, p. 75], though both these works avoid technical difficulties by substituting a parabola, but then incorrectly imply that Archimedes used such an approach for the parabola [3, p. 8] [12, p. 75]. The considerable length of the paper is a consequence of proving these results from basic principles. Unfortunately, it does not yet have a faithful English translation [60]; Heath's intent in [6] was to capture the modern flavor of Archimedes's works in order to make them more accessible. A generally faithful French translation, including the Greek text, is available [8].

The mechanical method does not seem to produce directly the area of a spiral, or even the area of a circle also computed by Archimedes so one might wonder how he first derived them. W.R. Knorr [41] has suggested that the writings of Pappus of Alexandria (fourth Century AD) indicate that Archimedes wrote an earlier version of *On Spirals* which used a different argument to compute the area of the spiral but then rejected it as inelegant (this approach is developed in the solution to Exercise 1). The object of the next section is to show how a natural extension of the mechanical method easily produces these results.

Weighing a spiral

The Method relies on a mechanical analogy by using a balance to compare objects. This requires a few simple assumptions and facts about the properties of a lever, which are developed (sometimes implicitly) in Archimedes's *On the Equilibrium of Planes I*, [6] [18, Chapter IX]. These can be summarized by

Assumption 1. Two objects will balance each other if the distances of their center of gravity to the fulcrum are inversely proportional to their weight. The center of gravity of an object lies on an axis of symmetry.

When only the weight of an object is relevant to an argument, I will place it on a *pan* suspended from the balance. The object and any of its sections will then be assumed to have their center of gravity at the point where the pan is suspended. I will also make extra assumptions not seen in Archimedes's works (however, see the solution to Exercise 4)

Assumption 2. A plane figure is composed of circular arcs with common center and each circular arc weighs the same as a line segment of equal length.

Exercise 1. What happens if you instead decompose plane figures into radii with common center?

I will first show how the mechanical method can be used to derive Archimedes's formula for the area of a circle given in *Measurement of the Circle,* Proposition 1 [58, Vol. 1, p. 317] (a similar method was used by Rabbi Abraham bar Hiyya (1070–1136), see [38] [53]).

Proposition 1. *Any circle is equal to a right–angled triangle in which one of the sides about the right angle is equal to the radius, and the base is equal to the circumference.*

Exercise 2. Explain why Proposition 1 is equivalent to the familiar formula: Area of a circle=πR^2.

Figure 1.

Suspend two pans on opposite sides of a balance and at equal distances to the fulcrum. On one pan, place a circle with center at A and radius AB, on the other place a line segment CD of length AB. By Assumption 2, the circle is composed of circumferences with center A and radius AE for any E lying on AB. For each such circumference, place a line segment FG perpendicular to CD, of length the circumference through E such that its endpoint F lies on CD and CF is equal to AE. By Assumption 2, the line segment FG is in equilibrium with the circumference through E. The resulting figure is a right triangle of height AB, base the circumference through B, and it balances a circle of radius AB, which is the statement of Proposition 1.

Exercise 3. Why is the resulting figure in this construction a triangle?

Exercise 4. Generalize the following heuristic from *The Method* [6, Supplement]: "…judging from the fact that any circle is equal to a triangle with base equal to the circumference and height equal to the radius of the circle, I apprehended that, in like manner, any sphere is equal to a cone with base equal to the surface of the sphere and height equal to the radius."

Figure 2.

Archimedes's definition of a spiral and its relevant components is given by [58, Vol. 2, p. 183]:

1. If a straight line drawn in a plane revolve uniformly any number of times about a fixed extremity until it return to its original position, and if, at the same time as the line revolves, a point move uniformly along the straight line, beginning at the fixed extremity, the point will describe a *spiral* in the plane.

2. Let the extremity of the straight line which remains fixed while the straight line revolves be called the *origin* of the spiral.

3. Let the position of the line, from which the straight line began to revolve, be called the *initial line* of the revolution.

Figure 3.

Proposition 2. *The area inside a spiral anywhere within its first revolution is one third the sector of a circle with center at the origin of the spiral, radius equal to the distance of the point describing the spiral to the origin, and angle equal to the angle between the line and the initial line.* (Archimedes gave areas for complete revolutions only, but his proof also applies to this case.)

Consider a spiral with origin A, initial line AB, and C the position of the point describing the spiral. Consider also a balance arm DE of length twice AC and let the midpoint F of DE be the fulcrum. On this balance suspend a pan from D and place the spiral region in the pan.

Figure 4.

By Assumption 2, the spiral region is composed of arcs GH for each G lying on AC, where H is the intersection of the circle with center A and radius AG and the spiral. Extend AH to intersect the circle of center A and radius AC at I. Consider a line segment JK of length equal to the arc CI and crossing DE at L such that JK and DE are perpendicular, L is the midpoint of JK, and FL is equal to AG. I claim that JK and the arc GH are in equilibrium. To see this note that, by Exercise 3, the length of an arc is proportional to its radius so that

$$\text{arc } GH : \text{arc } CI :: AG : AC,$$

and the result follows from the assumption that the arc CI has its center of gravity at D and from Assumption 1. Now extend the arc CI to intersect AB at M; then the arc CI is equal to the arc CIM minus the arc IM, and by the definition of spiral, IM is proportional to AH. Since the arc CIM remains constant in this argument, the second part of Exercise 3 shows that the arc CIM minus JK is proportional to FL, which means that the resulting figure is an isosceles triangle which balances the inside of the spiral.

Figure 5.

The exact same argument shows that the area between the spiral and the initial line that lies within the same sector balances the same isosceles triangle, but reversed so that its vertex lies on the fulcrum. The crucial step is to recall the following

Fact: The center of gravity of a triangle lies at the intersection of the medians, and the medians of a triangle intersect each other in a ratio of $2:1$.

The first part is suggested by the observation that a median divides a triangle into two triangles of equal weight and its proof is one of the main results of *On the Equilibrium of Planes I.* The second part is an easy exercise [15, §1.4] and follows from *On the Equilibrium of Planes I,* Proposition 15, generalized to trapezoids.

This shows that reversing the first triangle places the center of gravity twice as far from the fulcrum so the second triangle will balance twice the first. One concludes that the inside of the spiral weighs one half the outside of the spiral and thus one third of the sector of the circle, which is the statement of Proposition 2.

Exercise 5. Evaluate the area of the spiral using the procedure as for the circle, i.e., by only comparing weights placed on pans.

Exercise 6. Use the mechanical method to compute the center of gravity of a spiral region.

A modern translation

Figure 6.

The basic observation is that Assumption 2 extends Archimedes's method to polar coordinates. Consider a curve $r = f(\theta)$, in polar coordinates where, for simplicity, $f(r)$ is an increasing function, so there is an inverse function $\theta = g(r)$ (this notation is more convenient given the difficulties of Exercise 1). To compute the area of a region $\mathcal{A}$ lying inside the curve and having $0 \leq \theta \leq \Theta$, one partitions $\mathcal{A}$ into thin circular shells of width $h > 0$, as in Figure 6. Using the formula $\theta r^2/2$ for the area of a sector of angle θ and radius r, each shell has area $(\Theta - \theta)rh + R(r,h)$, where the error $R(r,h)$ is less than the area of the small shell element of area $(r+h)h[g(r+h) - g(r)]$, see Figure 6, and this is less than Crh^2, for some constant C, assuming that $g(r)$ is well behaved. It follows that, ignoring terms of order h^2, the area of each shell is $(\Theta - \theta)rh$, which is the length of the bottom arc of the shell multiplied by h. This shows why the first part of Assumption 2 holds. All these shells have area a linear function of h up to an error term of lower order, and form a disjoint union of $\mathcal{A}$, which shows why the second part of Assumption 2 holds. Letting $h \to 0$, it follows that the area of $\mathcal{A}$ is

$$\int_0^R (\Theta - \theta) r dr = \int_0^R [g(R) - g(r)] r dr \,.$$

The standard derivation of this formula uses the formula $r dr d\theta$ for the area element in polar coordinates

$$\text{Area of } \mathcal{A} = \iint_{\mathcal{A}} dx dy = \iint_{\mathcal{A}} r dr d\theta = \int_{r=0}^{R} \int_{\theta=g(r)}^{\Theta} r d\theta dr = \int_0^R [g(R) - g(r)]\, r dr \,.$$

A circle is simply $g(r) = 0$ which yields $2\pi \int_0^R r dr = \pi R^2$.

A spiral, in polar coordinates, is given by the equation $r = a\theta$, for some constant a, so can be written as $\theta = kr$, where $k = 1/a$. By the above, the area of the spiral is

$$\int_0^R (kR - kr) r dr = kR \int_0^R r dr - k \int_0^R r^2 dr = \frac{kR^3}{2} - k\,\frac{R^3}{3} = \frac{kR^3}{6} = \frac{1}{3}\,\frac{\Theta R^2}{2}\,,$$

where the term on the right is seen to be 1/3 the area of the sector of the circle of radius R and angle Θ, yielding Proposition 2.

Any proof of this formula is equivalent to evaluating such integrals. Archimedes evaluated $\int_0^R r^2 dr$ by decomposing it into Riemann sums and obtaining a closed form for the sum $1^2 + \cdots + n^2$. In Section 2 this integral is computed by realizing it as the moment of a triangle and evaluating this as its weight multiplied by the distance of its center of gravity from the fulcrum.

The Way of Archimedes

The Calculus Reform movement has emphasized experimentation over rigor in calculus education and has been criticized as a result [57]. To defend its position that physical problems should be used to discover mathematical results Harvard Calculus appeals to Archimedes and *The Method* [35, p. vii]

The Way of Archimedes: *Formal definitions and procedures evolve from the investigation of practical problems.*

This principle accurately represents the works of Archimedes, but a disparity arises in that Harvard Calculus postpones mathematical rigor indefinitely, so Archimedes's name should be among the last to be associated with such an endeavor. For example, the method of exhaustion used by Archimedes is essentially the ε–δ argument abandoned by Harvard Calculus, as B.L. van der Waerden writes [62, p. 220]:

> ... the estimations, which occur in the summing of infinite series and in limit operations, the 'epsilontics', as the calculation with an arbitrary small ε is sometimes called, were for Archimedes an open book. In this respect, his thinking is entirely modern.

Moreover, Archimedes held in contempt those who did not furnish proofs of their results. In the introduction to *On Spirals,* Archimedes reveals that he intentionally announced false theorems in order to expose some of his contemporaries [6]

> ... I wish now to put them in review one by one, particularly as it happens that there are two among them which [are wrong and which may serve as a warning to] those who claim to discover everything but produce no proofs of the same may be confuted as having actually pretended to discover the impossible.

Harvard Calculus fails miserably when measured against this Way of Archimedes. Apart from the passage quoted above, the word "theorem" appears in [35] only in the name "Fundamental Theorem of Calculus." Compare this with a standard calculus text [22] which lists 130 theorems in its index. Even more revealing, the only instance of the word "proof" I located in [35] was in Archimedes's introduction to the method quoted above and used in [35] to justify "The Way of Archimedes." In fact, this quote emphasizes that discovery of the answer to a problem leads to a *theorem* whose *proof* is facilitated by knowledge of the answer. My interpretation is not Calculus Reform but

Problem Solving: When faced with a problem, use *any* method that allows you to conjecture the answer, then find a rigorous proof.

A recent development: The second edition of [35] has taken a more moderate approach to Calculus Reform and now includes some complete proofs [35, 2nd Edition, p. 78] and the ε–δ definition of a limit [35, 2nd Edition, p. 128]. However, this new edition no longer includes "The Way of Archimedes."

Popular Misconceptions

It must be noted that the penultimate remark of the previous section paraphrases E.T. Bell [11, p. 31]: "In short he used mechanics to advance his mathematics. This is one of his titles to a modern mind: *he used* *anything and everything that suggested itself as a weapon to attack his problems."* However, strong opinions such as those expressed in [11] are fraught with danger, and it is instructive to include the continuation of this passage

> To a modern all is fair in war, love, and mathematics; to many of the ancients, mathematics was a stultified game to be played according to the prim rules imposed by the philosophically–minded Plato. According to Plato only a straightedge and a pair of compasses were to be permitted as the implements of construction in geometry. No wonder the classical geometers hammered their heads for centuries against 'the three problems of antiquity': to trisect an angle; to construct a cube having double the volume of a given cube; to construct a square equal to a circle.

This has since been discredited, see [24] [42] (better yet, look at original sources, e.g., as collected in [58, Vol. 1, Chapter 9]), and van der Waerden writes [62, p. 263],

> The idea, sometimes expressed, that the Greeks only permitted constructions by means of compasses and straight edge, is inadmissible. It is contradicted by the numerous constructions, which have been handed down, for the duplication of the cube and the trisection of the angle.

In particular, Archimedes trisected the angle with ruler and compass in Proposition 8 of *The Book of Lemmas* [6, p. 309], see [20] [31, Section 31]. The history of this misconception might prove an interesting subject for further study.

Unfortunately, it is only one of a number of popular misconceptions about the limitations of Greek science [60]. For example, Isaac Asimov (1920–1992) has written [5]

> To the Greeks, experimentation seemed irrelevant. It interfered with and detracted from the beauty of pure deduction... To test a perfect theory with imperfect instruments did not impress the Greek philosophers as a valid way to gain knowledge... The Greek rationalization for the "cult of uselessness" may similarly have been based on a feeling that to allow mundane knowledge (such as the distance from Athens to Corinth) to intrude on abstract thought was to allow imperfection to enter the Eden of true philosophy. Whatever the rationalization, the Greek thinkers were severely limited by their attitude. Greece was not barren of practical contributions to civilization, but even its great engineer, Archimedes of Syracuse, refused to write about his inventions and discoveries... to maintain his amateur status, he broadcast only his achievements in pure mathematics.

This passage is contradicted by numerous examples of Greek scientific experiments, for example, Eratosthenes's measurement of the earth [4]. Asimov may be excused for paraphrasing Plutarch's account of Archimedes in his *Life of Marcellus,* written circa 75 AD [50] [58, Vol. 2, p. 31]

> Yet Archimedes possessed so lofty a spirit, so profound a soul, and such a wealth of scientific inquiry, that although he had acquired through his inventions a name and reputation for divine rather than human intelligence, he would not deign to leave behind a single writing on such subjects. Regarding the business of mechanics and every utilitarian art as ignoble or vulgar, he

> gave his zealous devotion only to those subjects who elegance and subtlety are untrammeled by the necessities of life...

Despite Plutarch's ancient credentials he had no better insight into Archimedes's scientific contributions which contradict his story. The reader is already aware that *The Method* shows that physical considerations played an important role in Greek mathematics. But Asimov and Plutarch are completely refuted by Archimedes in *The Sand Reckoner* [6] [18]

> While examining this question I have, for my part tried in the following manner, to show with the aid of instruments, the angle subtended by the sun, having its vertex at the eye. Clearly, the exact evaluation of this angle is not easy since neither vision, hands, nor the instruments required to measure this angle are reliable enough to measure it precisely. But this does not seem to me to be the place to discuss this question at length, especially because observations of this type have often been reported. For the purposes of my proposition, it suffices to find an angle that is not greater than the angle subtended at the sun with vertex at the eye and to then find another angle which is not less than the angle subtended by the sun with vertex at the eye.
>
> A long ruler having been placed on a vertical stand placed in the direction the rising sun is seen, a little cylinder was put vertically on the ruler immediately after sunrise. The sun, being at the horizon, can be looked at directly, and the ruler is oriented towards the sun and the eye placed at the end of the ruler. The cylinder being placed between the sun and the eye, occludes the sun. The cylinder is then moved further away from the eye and as soon as a small piece of the sun begins to show itself from each side of the cylinder, it is fixed.
>
> If the eye were really to see from one point, tangents to the cylinder produced from the end of the ruler where the eye was placed would make an angle less than the angle subtended by the sun with vertex at the eye. But since the eyes do not see from a unique point, but from a certain size, one takes a certain size, of round shape, not smaller than the eye and one places it at the extremity of the ruler where the eye was placed...the width of cylinders producing this effect is not smaller than the dimensions of the eye.
>
> ...It is therefore clear that the angle subtended by the sun with vertex at the eye is also smaller than the one hundred and sixty fourth part of a right angle, and greater than the two hundredth part of a right angle.

The correct value of the angular diameter of the sun is now known to average about $34'$ [26, p. 95], i.e., the 159th part of a right angle. It is important to note that this shows not only that ancient Greeks frequently performed experiments, but that Archimedes dealt with experimental error and also compensated for the fact that the human eye is part of the observational instrument, thus anticipating scientists such as Hermann von Helmholtz (1821–1894) [34]. A translation and analysis of *The Sand Reckoner* is given in [60].

Answers to Exercises

Exercise 1. A naive approach leads to incorrect results, evidence of the dangers of using infinitesimals, and indicating why Archimedes did not consider his method to be rigorous. For example, taking the radii of a circle of radius R, with respect to the circumference, and reordering them to form a rectangle yields area

$2\pi R^2$. For a general figure, it's not even clear how to pick the radii. To make sense of what is going on, one regards radii as limits of sectors, i.e., infinitesimal triangles. In the case of the circle, this means that the weight of a radius, with respect to the circumference, is equal to *one half* its length. This can be loosely interpreted as the argument Archimedes used to compute the area of the circle [1]. In the general case, the following is justified:

Assumption 3. The weight of a radius is proportional to the square of its length.

In modern notation, this is simply

$$\iint_{\mathcal{A}} r dr d\theta = \int_{\theta=0}^{\Theta} \int_{r=0}^{f(\theta)} r dr d\theta = \frac{1}{2} \int_{0}^{\Theta} [f(\theta)]^2 d\theta \,,$$

where the radii have been chosen with respect to the unit circle. Given Assumption 3, one can compute the area of the spiral by using Pappus's argument [49, Book 4, Proposition 21], see also [32, p. 377] [42, p. 162].

Figure 7.

To compute the weight of a spiral region, take each radius of the spiral, starting from the final radius, and place a disk with diameter equal to this radius at height the current angle so the resulting figure is a cone. Similarly, for each radius of the sector place a disk with diameter equal to this radius at height the current angle, resulting in a cylinder with the same base and height as the cone.

Since Euclid's Proposition 2 of Book 12 proves that "circles are to one another as the squares on the diameter," Assumption 3 shows that the ratio of the weight of the spiral region to the weight of the sector is the same as the ratio of the volume of the cone to the volume of the cylinder. But Euclid's Proposition 10 of Book 12 proves that the volume of a cone is one third the cylinder with same base and height, so the spiral weighs one third of the sector, which is the statement of Proposition 2. (Note that equilateral triangles could have been used instead of circles resulting in a pyramid whose volume is easier to compute.)

Knorr [41] comments that this appeal to three dimensional figures might have been considered inelegant by Archimedes as it uses volumes to compute areas. On the other hand, reversing this argument and using the evaluation above shows that the volume of a cone can be computed by the mechanical method, a result which does not appear in *The Method.*

Exercise 2. In modern notation, Archimedes's formulation of Proposition 1 is Area of circle of radius $R = \int_0^R 2\pi r dr$, for the integral represents the area of a right triangle with base R and height $2\pi R$.

Exercise 3. This is equivalent to the fact that the length of an arc of fixed angle is proportional to its radius. In particular, π exists, see [46] [60]. The proof is similar to [23, Book 12, Proposition 2] cited in Exercise 1, and is implicit in Archimedes's *Measurement of the Circle.* Similarly, the length of an arc of fixed radius is proportional to its angle.

Exercise 4. By analogy with Assumption 2, consider a sphere as being composed of spherical shells centered at the center of the sphere, where each shell weighs the same as a circle of equal area. The justification follows exactly as in Proposition 2: Consider two pans suspended at equal distances from the fulcrum of a balance. On one pan, place a sphere of center A and radius AB and on the other a line CD of length equal to AB. For each E on AB there is a spherical shell passing through E, and consider a circle of area equal to this spherical shell with center at F lying on CD, where CF equals AE, and such that the circle is perpendicular to CD. The resulting figure is a cone with base the area of the sphere and height the radius of the sphere; since it balances the sphere, the claim is justified.

The similarity of this argument to the one of Proposition 1 suggests that Archimedes may have been implicitly aware of the ideas of this paper. Moreover, the reader may verify that the heuristic of this exercise and its justification directly generalize to higher dimensions (a different generalization is given in [19]):

Proposition 3. *The volume of an n–dimensional ball is equal to the volume of a cone whose base has $n-1$–dimensional volume equal to the $n-1$–dimensional volume of the boundary of the ball and height equal to the radius of the ball.*

Figure 8.

Exercise 5. The procedure when applied to the spiral, yields a section of a parabola. The general formula for such areas was computed by Archimedes in *The Quadrature of the Parabola,* and in this case it states that the resulting area is four–thirds the triangle with same base and height as the section of the parabola. Since the height and base are equal to the final radius and half the final radius, respectively, Proposition 2 follows.

Exercise 6. Further extensions of Archimedes's method could be a subject for investigation. As Archimedes wrote in *The Method* [6, Supplement, p. 13],

> I deem it necessary to expound the method partly because I have already spoken of it but equally because I am persuaded that it will be of no little service to mathematics; for I apprehend that some, either of my contemporaries or of my successors, will, by means of the method when once established, be able to discover other theorems in addition, which have not yet occurred to me.

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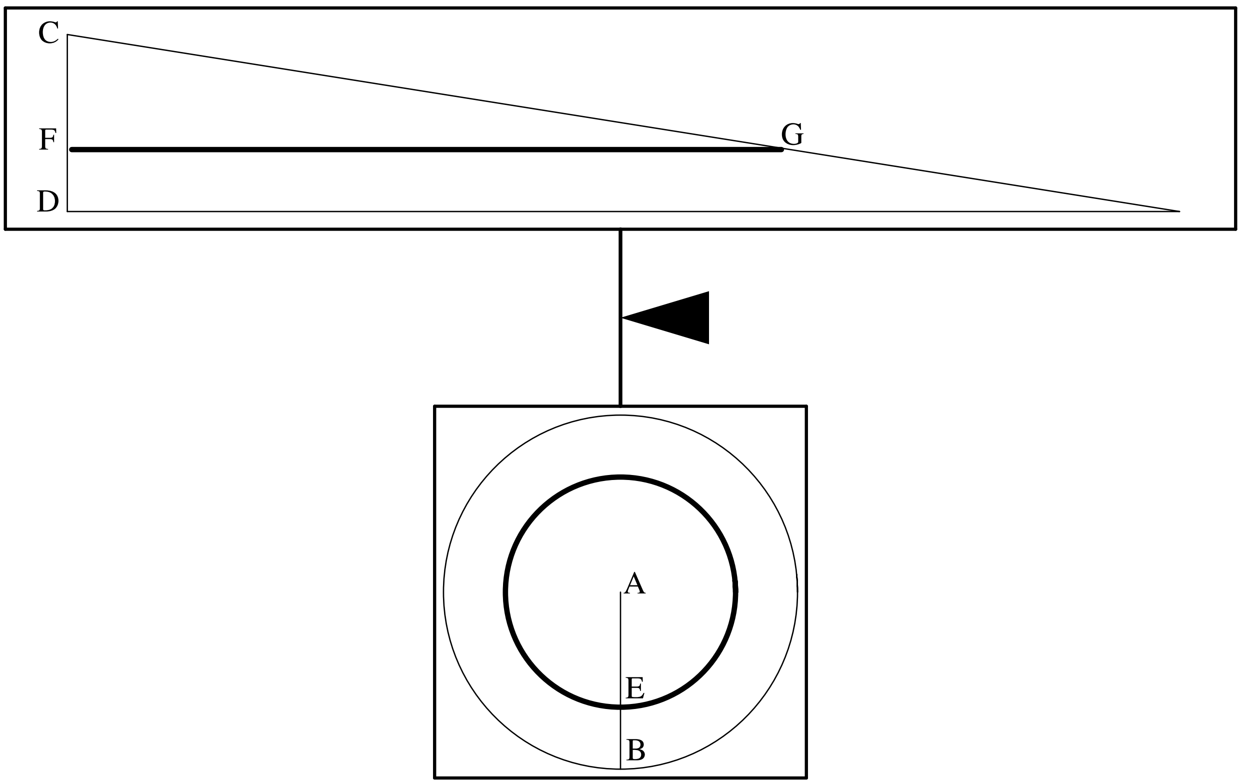
Figure 1.



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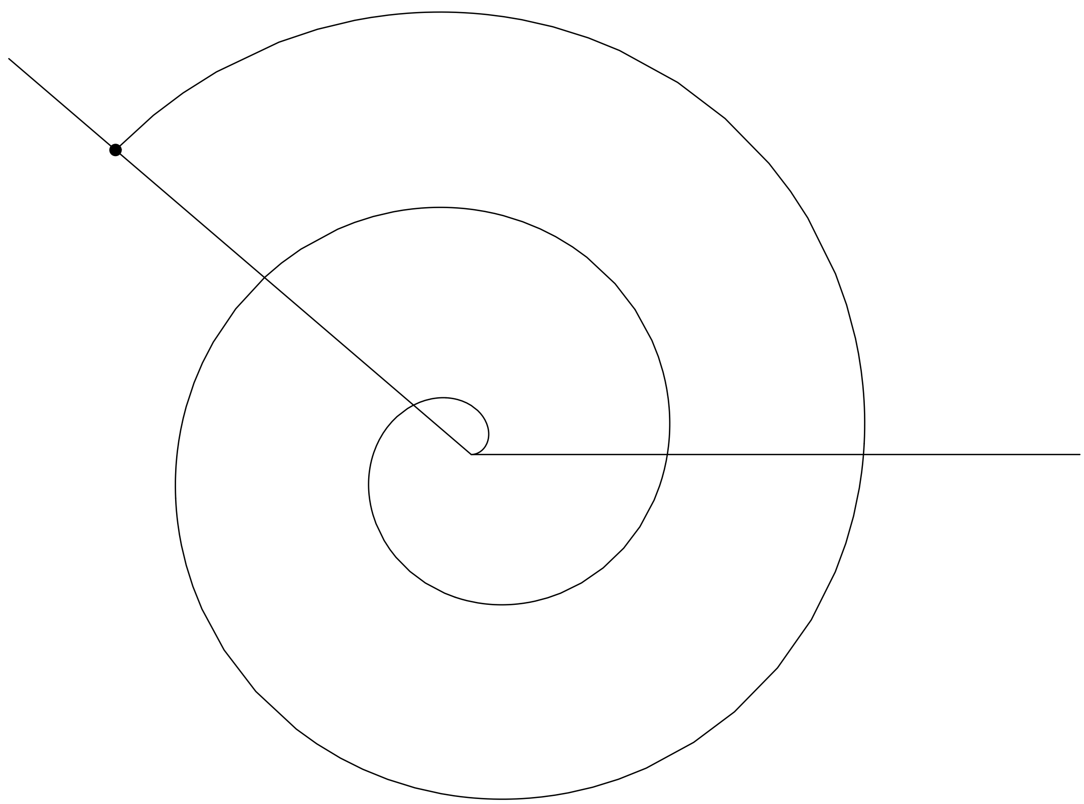

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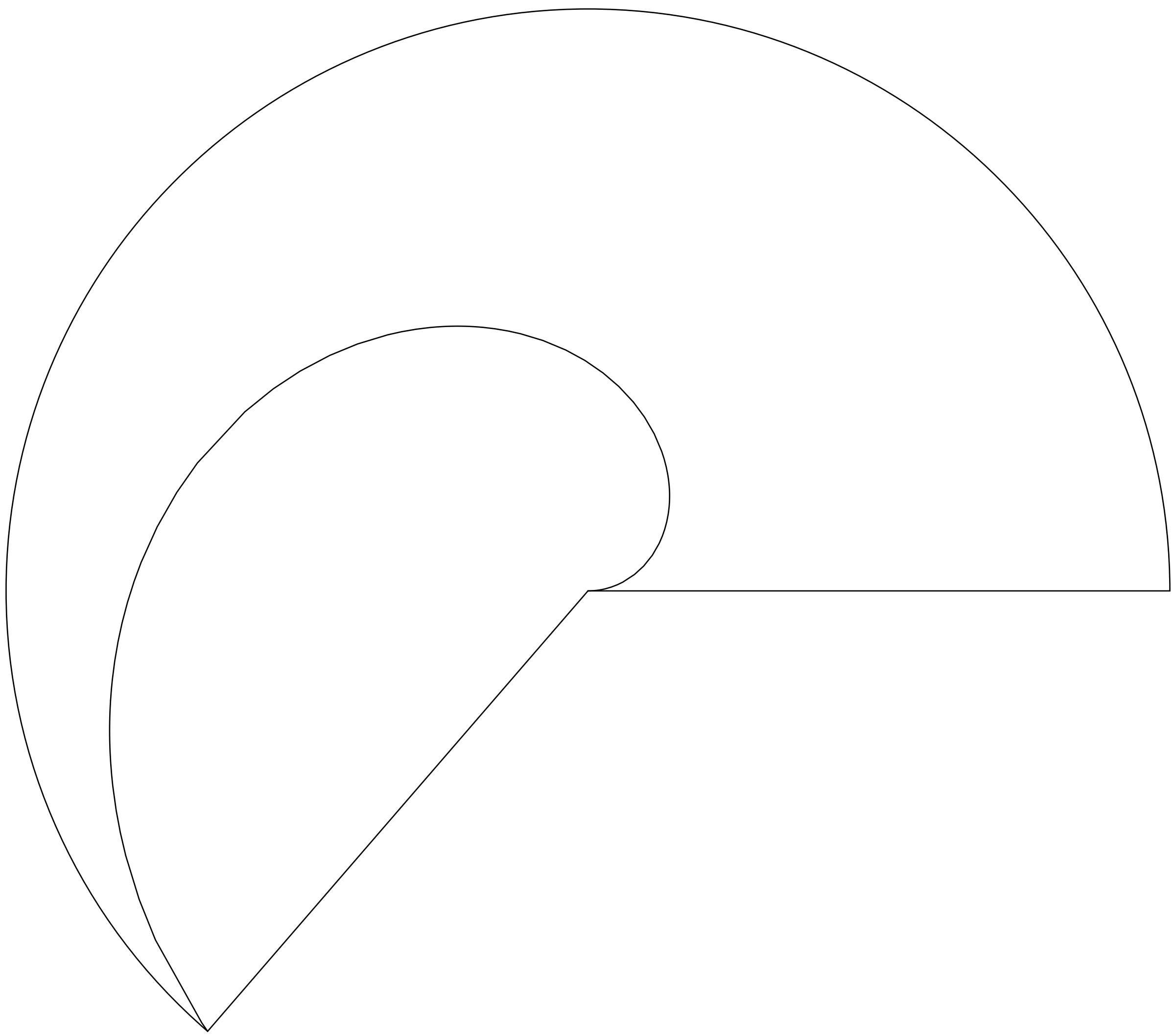

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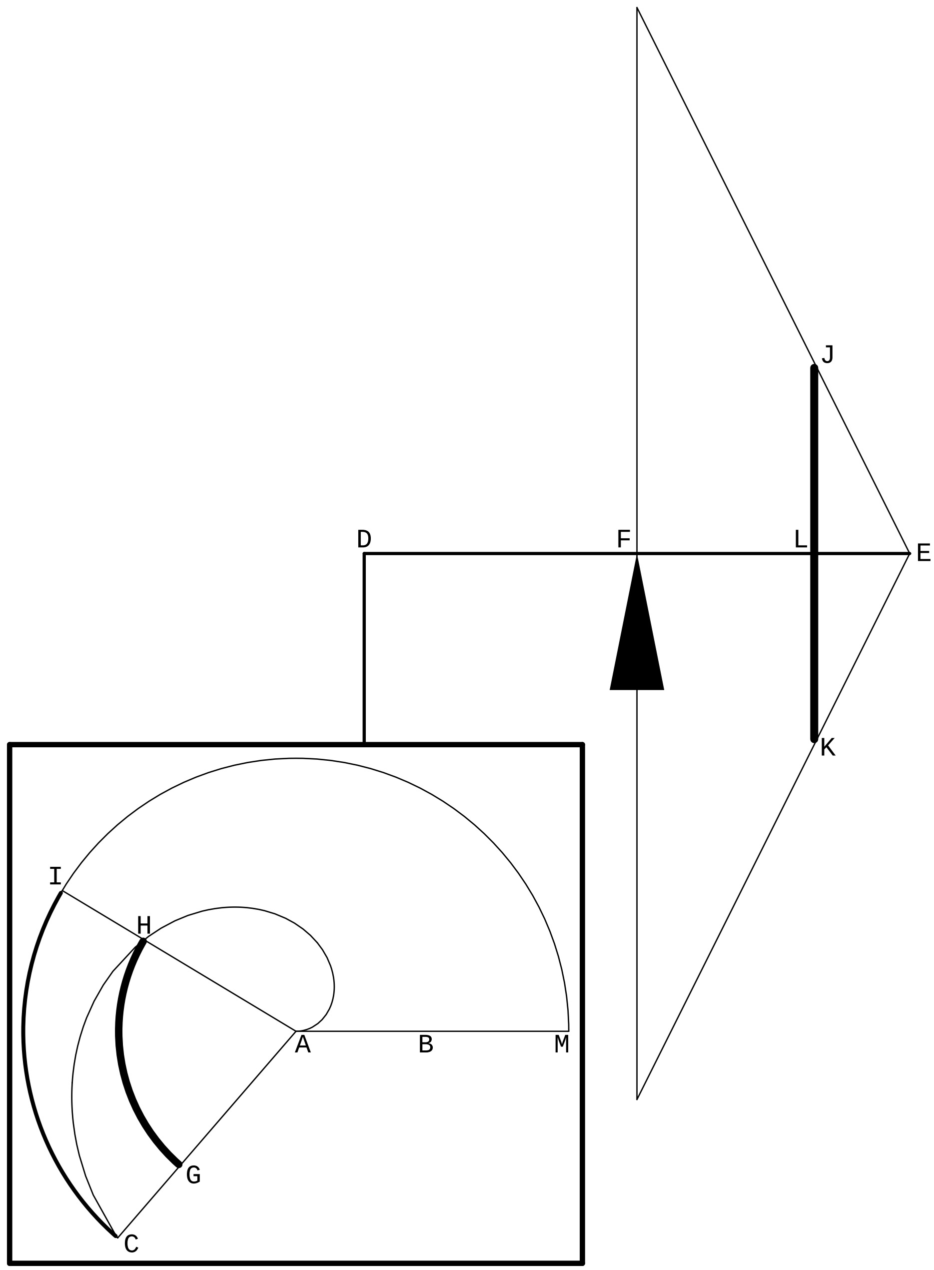

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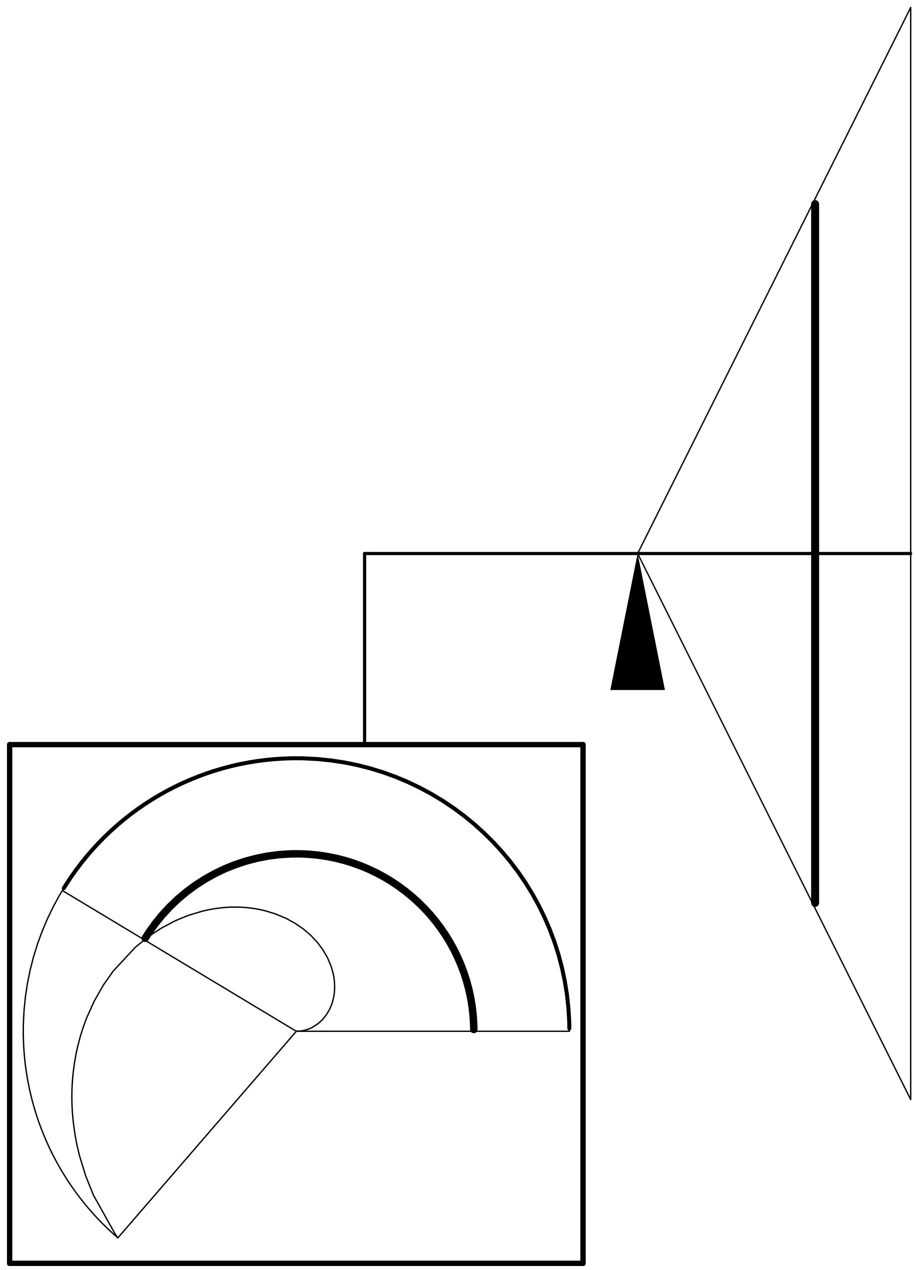

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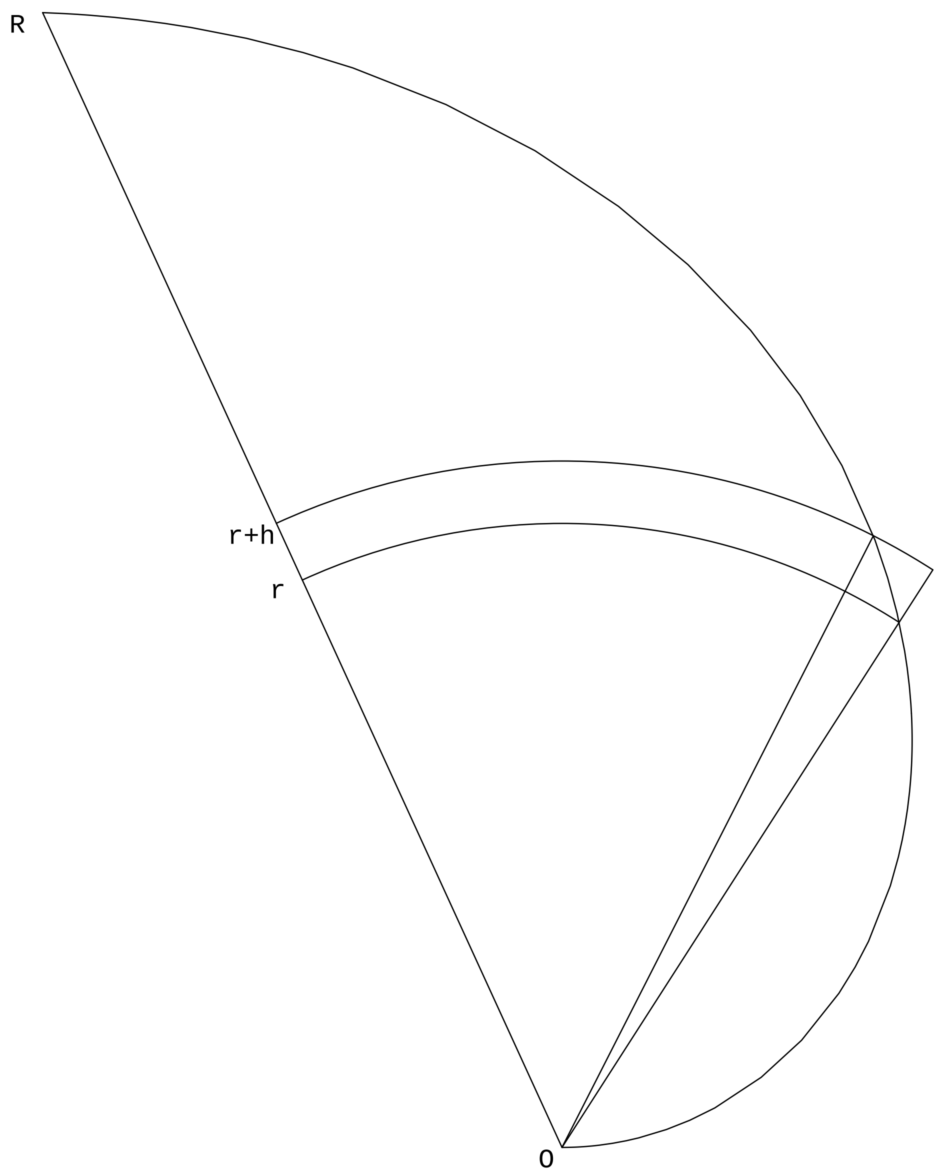

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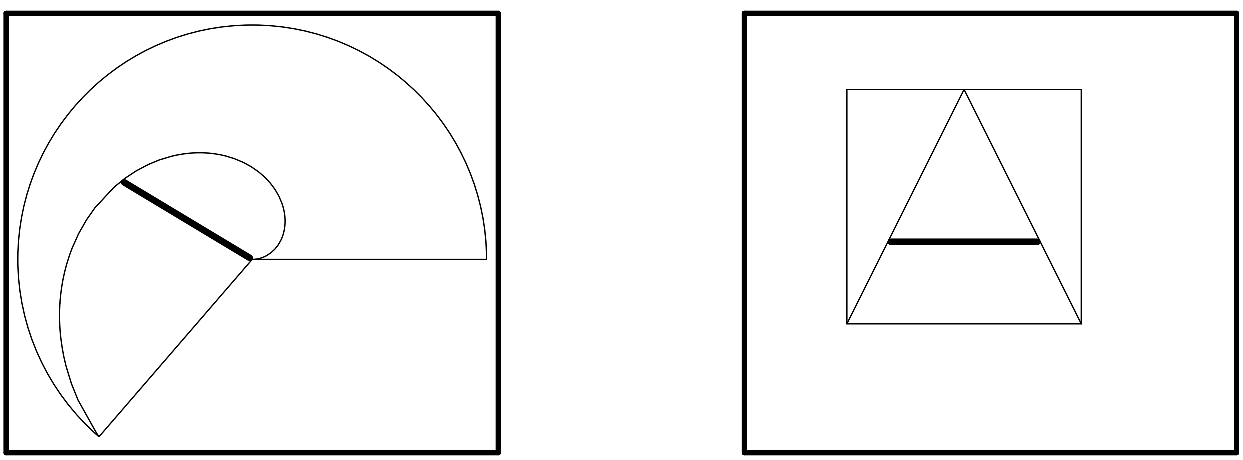

Figure 8.